Weight derivation in analytic hierarchy process using data envelopment analysis

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Abstract

Analytic Hierarchy Process (AHP) is known for its great flexibility and wide applicability. Because of ease of use, AHP has been studied extensively and used in all applications related to multi criteria decision making (MCDM) in areas like manufacturing, textile and fashion, food, telecommunication, education, hospitals, government, construction, pharmaceuticals, marketing, retailing, supply chain management, logistics, human resource selection, course design and evaluation and many more since of its inception. AHP converts qualitative opinion of experts into quantitative values and generates a comparison matrix. In this paper we have integrated Data Envelopment Analysis to generate local weights of alternatives from pairwise comparison matrix used in AHP. Multiple expert judgements have been considered for weight determinations of criteria and sub-criteria. This process does not suffer rank reversal when an irrelevant alternative(s) is added or removed.


Article History
Received: 11 April 2018
Revised: 22 June 2018
Accepted: 22 June 2018

Keywords
Aggregation,
Analytic Hierarchy Process
Data Envelopment Analysis
Linear Programming
1. Introduction

In today’s world, problem solving consists of a constant increase in complexity and unfamiliar situations to tackle (Ramanathan & Ganesh, 1994; Luo, 2003; Singh, 2011). For these crucial situations, decision makers—individuals, organizations and even nations choose mathematical tools (Suwignjo et al., 2000; Emrouznejad and Thanassoulis, 2005). This is because mathematical decisions use quantitative techniques that cut through information and utilize these powerful tools and techniques to understand the situation in details and provide optimal solutions.

Making a decision which has long-term implications requires a thorough understanding of likely or possible future situations and also the ability to balance a large number of controllable and uncontrollable parameters (Patel & Godwin, 2016). A good decision must, therefore, be a predictor of correct outcomes for a long period of time.

Consumers postpone their consumption to meet their future needs. This is known as savings. Savings have to channelize in a proper way so that it multiplies and gives the targeted yield. This is called investments.

Investors normally expect returns in proportion to the risk they undertake in investment. The time horizon to stay invested is also reasonable (Smith, 1990).

The investment path depends on the goals which investors want to pursue. However the investor is always concerned with the following factors which influences their decisions:

1. Expected Return
2. Risk
3. Liquidity
4. Tax Aspects
5. Time Horizon

The choice of investments options viz. Equity, Debentures, Bonds, Treasury Bills will revolve round the above factors. Whether the investors make direct investment in any of the above mentioned instruments or comes through the mutual fund, the above factors are really very critical for all kinds of investors.

Expected Returns - Returns depends on the risk-free return and perception of the risk associated with the investment. Risk free return is the return which the investor will get for parting with the liquidity of the funds. However the
perceived default risk is zero. However, as the risk level goes up, the expectation of the investors also goes up. The actual returns might vary depending on the performance of the investment asset. Inflation impacts the expected return. Higher inflation would mean high interest and high expected return.

Investors usually lose money because they aim for returns which are far from the reality. A small investor is the least informed person and, therefore, is unable to respond timely to the market movements.

Risk - It means the variability of return. The swing could be positive or negative. If the risk is high, it will be difficult to predict the returns. The investors may or may not be able to reach the expected return. Some risk can be eliminated by diversification while some risk cannot be eliminated at all. These are known as market risk. Investor will strive to increase return for a given level of risk or reduce risk for a given level of return. Risk is measured in terms of beta. A high beta indicates high risk and vice versa.

Depending on the risk appetite of the investor, financial security may be chosen for investment.

Liquidity - The investor is always concerned about the liquidity aspect. While the maturity date of the investment could be anything, but is it possible for the investor to come out of the investment commitment before maturity date? This aspect is called Liquidity. The investor would always like to use this option in case of any emergency or any unforeseen circumstances.

Liquidity is the ability to convert the financial security into cash. Investors usually want to keep at least some portion of investment in such assets which are highly liquid.

Tax Aspects - The investor is always concerned to reduce the tax liability in a legal and permissible way. Therefore, the investment is greatly influenced by Tax Liability / Tax Benefits. If the tax rates are higher in the economy, then this factor becomes more crucial. The investor is concerned with post tax returns and hence tax aspects impacts in a big way.

Time Horizon - Time period for which the investment has to be made is a critical factor to be considered. Time Horizon determines how much investment is to be made to meet the goal. Investors normally do not prefer to stay invested for a long time.

This study aims to examine the above factors and establish ranking for the above factors using DEAHP.
2. Methodology

2.1 Analytic Hierarchy Process

The main source of complexity in decision making is the presence of actors and stakeholders with different and contradicting interests and objectives with other actors. For a decision to be successful, the objective, views and interest of all stakeholders must be taken into consideration. As the current global situation has become uncertain, it is important for policy makers and management to understand the broader consequences of their decisions. This is because such decisions are likely to have implications on the future situation. A good decision must therefore be a predictor of correct outcomes for a long period of time (Saaty, 1980).

Organizations deploy the AHP technique in high stake situations where human resource decisions and judgments could have long lasting consequences of their organizations. Today, decision making is not as complex as it used to be. It is more systematic and scientific. A decision choice is disintegrated into structures involving risks, benefits, costs, opportunities and threats. Through the use of AHP, management is enabled to select the most feasible solution. The other advantage of the AHP is that it can be used in group decision making.

It operates a complete model, signifying the relations of importance dominance and preference among the aspect of the problem. The AHP is structured as a hierarchy where a decision is met by the decomposition of the goal into the most general and most likely and easily controlled factors. It requires an assessment of various criteria and the evaluation to achieve the relative ranking of the alternatives with respect to the problem.

This is further compounded when there are several or more experts whose opinions need to be incorporated in the decision making. While applying the AHP, users have to recognize that the methodology has a deficiency of quantitative information and thus it depend on experts qualities like intuition, experience and judgment using a scale of relative importance to obtain weights that will help them to order the alternatives in such a way that the solution is realized objectively. This solidifies the basis of the multi-criteria decision making in AHP.

The multi-criteria decision making in AHP is applicable in various situations of decision making. It is the most widely used multi criteria-decision making tool (Chan & Lunn, 1991). Its application has assisted decision making in
management and in many such fields, including economic planning, selection of
the best production techniques, conflict resolutions and resource optimization
models.

The uniqueness of the AHP methodology is its flexibility and ease of
integration into various techniques including linear programming.

Another feature of the AHP is in determining choices. The determination
of choices implies the availability of a set of choices from which a decision maker
can choose. When a choice is made between different alternatives, it implies that
other choices are left.

Successfully, AHP has been employed in a number of choice decisions.
Indeed, the AHP is broader than just being a technique for determining choices.
First and foremost, AHP is able to structure complexities. Saathy (1980) came up
with the hierarchical structuring of complexities into a homogeneous cluster of
factors. In measuring on a ratio scale, he was convinced that the ratio scale
priorities of elements measure the factors that are comprised in the hierarchy.
The hierarchy in AHP can be shown as in Fig 1,

**Figure1: The Hierarchical Structure**

```
Goal

Criterion 1  Criterion 2  ...  Criterion P

Sub criterion 1
Sub criterion 1L

Sub criterion 2
Sub criterion 2M

Sub criterion P
Sub criterion PN

Alternative 1  Alternative 2  Alternative 3  ...  Alternative Q
```

The second use of the AHP is the resolution of the choice problems in
multi criteria environment. In that mode, its methodology includes comparison
of objectives and alternative in natural pairwise manner. It converts individual preferences into ratio scale weights that are combined into linear additive weights for the associated alternatives.

This is such that a consistency bound is established for a specific comparison pairwise matrix. Priorities however, should be unique and not one of many possibilities. They must also capture the dominance of the order and be expressed in the judgment of the comparison matrix. The scale ranges from $1/9$ for ‘least valued than’ to $1$ for ‘equal’ and $9$ for ‘absolutely more important than’ covering the entire spectrum of the comparison.

The AHP is also used in prioritization or evaluation. Prioritization is the determination of relatively important alternatives among a give set of alternatives, as opposed to selecting an alternative as a choice of application. While prioritizing alternatives the order, the interval and the ratios of the resulting priorities are of interest in addition to knowing which alternative has the highest priority. As the priorities derived from AHP are ratios measures, these priorities can be used in selecting a combination of alternatives or in allocating resources.

The inconsistency level of the decision maker’s pair wise comparison is checked.

As a mathematical model AHP is a compositional methodology that synthesizes a decision maker’s preference judgment for each of the decision alternatives under each criterion within a decision hierarchy. This type of assessment is done by a pairwise comparison which responds appropriately to a posed question in order to elicit judgments.

The fundamental scale originally proposed by Saaty for the AHP consisted of the words ‘equal’, ‘weak’, ‘strong’, ‘very strong’, and ‘absolute’ which are used. Saaty proposed representing the intensity of these words with ratios of $1, 3, 5, 7$ and $9$ respectively, with even integers $2, 4, 6, 8$ being used for intermediate judgment such as $6$ for between ‘strong’ and ‘very strong’.

2.2 Eigenvector Method (EVM)

This approach is based on grounds that assume small perturbations of the element $a_{ij}$ of matrix $A$ from the perfect ratios of $w_i/w_j$. This leads to small
perturbations of the eigenvalues of the comparison matrix $A$. Saaty proves that the principal eigenvector of $A$ can be used as the desired priority vector. EVM is based on solving the following equation.

$$Aw = \lambda w, e^T w = 1$$

The principal eigenvector $\lambda_{max}$ of $A$ is determined by solving the characteristic equation,

$$\text{det}(A - \lambda_{max}I) = 0$$

Where $I$ is the identity matrix.

Then, using the value of $\lambda_{max}$, the eigenvector $w = (w_1, w_2, \ldots, w_n)$ is found out from the set of simultaneous linear equations:

$$(A - \lambda_{max}I)w = 0$$

2.3 Consistency Check

AHP tolerates an inconsistency ratio of less than equal to 10% taking into account the different units of criteria and goals to be compared. To check the inconsistency, the consistency index ($CI$), Random consistency Index ($RI$), Consistency Ratio ($CR$) are calculated as:

$$CI = \frac{\lambda_{max}}{(n - 1)}$$

$$RI = 1.98 \times \frac{(n - 2)}{n}$$

$$CR = \frac{CI}{RI}$$

Where $\lambda_{max} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} w_j$

2.4 Summary of the Methodology behind AHP

Analysis- where the problem is defined and decomposed into a list of related elements and sub-problems (structure the complexity) this helps decision makers to focus on understanding each element according to its importance and effect on the overall process.

Hierarchy - where a structure is organized into levels of criteria and sub-criteria in relation to a given goal. A matrix of pairwise comparisons at each
level is made. The lowest level of different alternatives is compared with respect to each element.

Process- Mathematical methods are used to obtain weights at each level of the hierarchy from the goal to the alternative. A consistency ratio to check consistency of the judgments is used. To determine the importance of a criterion against another a scale is applied. This scale quantifies the degree of importance of one element to another, such as a criterion against a criterion, or sub-criterion against other sub-criterion.

The process of developing a model to solve investment problem is as follows:

i. Identifying decision problems.

ii. Listing every evaluation element.

iii. Setting up hierarchical relationship.

iv. Pairwise comparison.

v. Establishing pairwise comparison matrix.

vi. Calculating priority weights using different methods

vii. Find the consistency check.

2.5 Data Envelopment Analysis (DEA)

Constant Return to Scale

Suppose there are \( n \) Decision Making Units, \( \text{DMU}_j \) \((j = 1, \ldots, n)\), and each DMU using \( m \) inputs \( x_j = (x_{1j}, \ldots, x_{mj}) \) produces \( s \) outputs \( y_j = (y_{1j}, \ldots, y_{sj}) \). The following CCR model assesses the efficiency score of \( \text{DMU}_{0e(1\ldots n)} \), unit undervaluation:

\[
\begin{align*}
\max \theta & = \sum_{r=1}^{s} u_r y_{r0} \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{i0} & = 1 \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} & \leq 0 \quad j = 1, \ldots, n \\
v_i & \geq 0 \quad i = 1, \ldots, m \\
u_r & \geq 0 \quad r = 1, \ldots, s
\end{align*}
\]
where $v_i$ and $u_r$ are the weights of $i^{th}$ input and $r^{th}$ output, respectively. The CCR model involves $n$ constraints in common, $\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0; j = 1, ..., n$, and its feasible region differ only from the first constraint, $\sum_{i=1}^m v_i x_{io} = 1$.

The first constraint of model (1), which is called normalization condition, ensures that the weights are relative.

Let $(\theta^*, \nu^*, \mu^*)$ be the optimal solution of model (1). The following definition partitions all DMUs into two sets: efficient and inefficient. DMU$_o$ is CCR-efficient if and only if $\theta^* = 1$ and there exists at least one strictly positive optimal solution $(\nu^*, \mu^*)$. As a result, DMU$_o$ is CCR-inefficient if either $\theta^* < 1$ or $\theta^* = 1$ and for every optimal solutions there exists at least one zero weights. This model represents constant return to scale and is denoted as CCR model.

### 2.6 Linking of AHP and DEA

Both AHP and DEA are versatile tools in their own field. DEA has traditionally found in measurements of the performance of DMUs whereas AHP has been widely used in multi criteria decision making (MCDM). When we withdraw a criterion or insert a new criterion in AHP, there is a chance of rank reversal. It may also violate the property of element dominance. To avoid this, an AHP problem is linked with DEA (Ramanathan, 2006; Singh & Agrawal, 2014) as follows:

<table>
<thead>
<tr>
<th>Traditional AHP View</th>
<th>Proposed DEA View</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="#" alt="Table" /></td>
<td><img src="#" alt="Table" /></td>
</tr>
</tbody>
</table>

A list of applications using integrated approach of DEA and AHP can be found in the paper of William Ho and Xin Ma (2018).
3. Numerical Example

A sample of ten experts were asked to compare pair wise all ten questions in a scale of 1 to 9. The pairwise comparison matrix taking geometric mean is shown in Table 1.

Table -1 Pairwise Comparison Matrix

<table>
<thead>
<tr>
<th></th>
<th>Expected Return</th>
<th>Risk</th>
<th>Liquidity</th>
<th>Tax Aspects</th>
<th>Time Horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Return</td>
<td>1</td>
<td>2.773</td>
<td>2.289</td>
<td>1.644</td>
<td>1.260</td>
</tr>
<tr>
<td>Risk</td>
<td>0.361</td>
<td>1</td>
<td>0.794</td>
<td>1</td>
<td>1.022</td>
</tr>
<tr>
<td>Liquidity</td>
<td>0.437</td>
<td>1.260</td>
<td>1</td>
<td>1.387</td>
<td>0.669</td>
</tr>
<tr>
<td>Tax Aspects</td>
<td>0.608</td>
<td>1</td>
<td>0.721</td>
<td>1</td>
<td>0.843</td>
</tr>
<tr>
<td>Time Horizon</td>
<td>0.794</td>
<td>0.979</td>
<td>1.494</td>
<td>1.186</td>
<td>1</td>
</tr>
</tbody>
</table>

Based on Eigen Vector Method, $\lambda_{max}$, consistency index (CI), random consistency index (RI), consistency ratio (CR) and weights of all are calculated shown in Table 2. It can be observed that the comparison between risk and time horizon is, $a_{ij} > 1$, which implies the weight of risk be more than or equal to the weight of time horizon. But in this case the weight of risk (0.150) is less than the weight of time horizon (0.205). With the same set of data, the weights using DEAHP is presented in table 3. From this we can conclude that investors give weightage to expected return, risk, liquidity, tax aspects and time horizon 0.243, 0.197, 0.205,0.162 and 0.205 respectively.

Table 2 Weights and Consistency check based on Eigen Vector Method

<table>
<thead>
<tr>
<th></th>
<th>Expected Return</th>
<th>Risk</th>
<th>Liquidity</th>
<th>Tax Aspects</th>
<th>Time Horizon</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Return</td>
<td>1</td>
<td>2.773</td>
<td>2.289</td>
<td>1.644</td>
<td>1.260</td>
<td>0.321</td>
</tr>
<tr>
<td>Risk</td>
<td>0.361</td>
<td>1</td>
<td>0.794</td>
<td>1</td>
<td>1.022</td>
<td>0.150</td>
</tr>
<tr>
<td>Liquidity</td>
<td>0.437</td>
<td>1.260</td>
<td>1</td>
<td>1.387</td>
<td>0.669</td>
<td>0.167</td>
</tr>
<tr>
<td>Tax Aspects</td>
<td>0.608</td>
<td>1</td>
<td>0.721</td>
<td>1</td>
<td>0.843</td>
<td>0.156</td>
</tr>
<tr>
<td>Time Horizon</td>
<td>0.794</td>
<td>0.979</td>
<td>1.494</td>
<td>1.186</td>
<td>1</td>
<td>0.205</td>
</tr>
<tr>
<td>$\lambda_{max} = 5.088$</td>
<td>CI=0.022</td>
<td>RI=1.188</td>
<td>CR=0.019</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 3  Weights based on DEAHP method

<table>
<thead>
<tr>
<th></th>
<th>Output 1</th>
<th>Output 2</th>
<th>Output 3</th>
<th>Output 4</th>
<th>Output 5</th>
<th>Dummy Input</th>
<th>Efficiency</th>
<th>weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Return</td>
<td>1</td>
<td>2.773</td>
<td>2.289</td>
<td>1.644</td>
<td>1.260</td>
<td>1</td>
<td>1.000</td>
<td>0.243</td>
</tr>
<tr>
<td>Risk</td>
<td>0.361</td>
<td>1</td>
<td>0.794</td>
<td>1</td>
<td>1.022</td>
<td>1</td>
<td>0.811</td>
<td>0.197</td>
</tr>
<tr>
<td>Liquidity</td>
<td>0.437</td>
<td>1.260</td>
<td>1</td>
<td>1.387</td>
<td>0.669</td>
<td>1</td>
<td>0.843</td>
<td>0.205</td>
</tr>
<tr>
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<td>1</td>
<td>0.721</td>
<td>1</td>
<td>0.843</td>
<td>1</td>
<td>0.669</td>
<td>0.162</td>
</tr>
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<td>Time Horizon</td>
<td>0.794</td>
<td>0.979</td>
<td>1.494</td>
<td>1.186</td>
<td>1</td>
<td>1</td>
<td>0.794</td>
<td>0.193</td>
</tr>
</tbody>
</table>

4. Conclusion

Data envelopment analysis (DEA) has been proposed in this paper for deriving weights from the judgement matrix of the analytic hierarchy process. If the decision maker is not able to decide whether one alternative is better than other precisely, the problem of element dominance or rank reversal or both may arise. Since DEA is able to calculate true weights, we employed DEA to derive weights in pairwise comparison matrix obtained from AHP. The weights we found for expected return (0.243) is more and the weights of tax aspects (0.162) is less.

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